Pf (of derivate Formula) $x_n = f(x_1, \ldots, x_{n-1})$

Apply chain rule to compute partial $\frac{\partial f}{\partial x_i}$

$$\frac{\partial F}{\partial x_i} = \frac{\partial F}{\partial x_i} \cdot \frac{\partial x_i}{\partial x_i} + \frac{\partial F}{\partial x_n} \cdot \frac{\partial x_n}{\partial x_i}$$

$$\frac{\partial F}{\partial x_i} = \frac{\partial F}{\partial x_i} \cdot \cancel{\frac{\partial x_i}{\partial x_i}}^{1} + \frac{\partial F}{\partial x_n} \cdot \frac{\partial f}{\partial x_i} \rightarrow \frac{\partial f}{\partial x_i} = -\frac{\partial F / \partial x_i}{\partial F / \partial x_n}$$

Ex. compute $x^3 + y^3 + z^3 = 6xyz + 1$ $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Sol:

$x^3 + y^3 + z^3 = 6xyz + 1$ iff $x^3 + y^3 + z^3 - 6xyz - 1 = 0$

use $F(x, y, z) = x^3 + y^3 + z^3 - 6xyz - 1$

$$\frac{\partial F}{\partial x} = 3x^2 - 6yz, \quad \frac{\partial F}{\partial y} = 3y^2 - 6xz, \quad \frac{\partial F}{\partial z} = 3z^2 - 6xy$$

$$\frac{\partial z}{\partial x} = -\frac{\partial F / \partial x}{\partial F / \partial z} = -\frac{3x^2 - 6yz}{3z^2 - 6xy}$$

$$\frac{\partial z}{\partial y} = -\frac{\partial F / \partial y}{\partial F / \partial z} = \frac{3y^2 - 6xz}{3z^2 - 6xy}$$

Gradient and optimization

Def: The gradient of a function $f(x_1, x_2, \ldots, x_n)$ is

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_n} \right\rangle$$

$$\frac{\partial f}{\partial t_i} = \nabla f \cdot \frac{\partial \vec{x}}{\partial t_i} \qquad \vec{x} = \langle x_1(t_1 \ldots t_n), x_2(t_1, \ldots, t_n), x_3(t_1 \ldots t_n) \rangle$$

$$\frac{\partial \vec{x}}{\partial t_i} = \left\langle \frac{\partial x_1}{\partial t_i}, \frac{\partial x_2}{\partial t_i}, \frac{\partial x_n}{\partial t_i} \right.$$

$$D_u f(\vec{p}) = \lim_{h \to 0} \frac{f(\vec{p} + h\vec{u}) - f(\vec{p})}{h} \quad , \quad D_u f(\vec{p}) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0)$$

Consider $g(t) = f(\vec{p} + h\vec{u})$

$g(h) = f(p_1 + hu_1, p_2 + hu_2, p_n + hu_n)$

$$\frac{dg}{dh} = \nabla F \cdot \frac{\partial f}{\partial h}$$

$$= \nabla F \cdot \langle u_1, u_2, \dots u_n \rangle = \nabla F \cdot \vec{u}$$

$D_u f(\vec{p}) = \nabla f_{(p)} \cdot \vec{u}$

Ex. Lets compute the $D_{\vec{u}} f(\vec{p})$ for $f(x,y,z) = 4y\sqrt{x}$ $p = \langle 4, 1 \rangle$

$\vec{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$$\nabla f = \langle 2x^{-\frac{1}{2}} y, 4x^{\frac{1}{2}} \rangle \ , \ \nabla f_{(p)} = \langle 2(4)^{-\frac{1}{2}}(1), 4(\tfrac{1}{2})^{\frac{1}{2}} \rangle = \langle 1, 8 \rangle$$

$$D_u f(\vec{p}) = \langle 1, 8 \rangle \cdot \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle = \frac{1}{\sqrt{2}} - \frac{8}{\sqrt{2}} = -\frac{7}{\sqrt{2}}$$

Ex. Compute ∇F for $f(x,y,z) = \frac{xz}{y+z}$

sol:

$$\frac{\partial f}{\partial x} = \frac{z}{y+z} \quad , \quad \frac{\partial f}{\partial y} = -\frac{xz}{(y+z)^2}$$

$$\frac{\partial f}{\partial z} = \frac{(x)(y+z) - (1)(xz)}{(y+z)^2} = \frac{xy}{(y+z)^2}$$

$$\nabla f = \langle \frac{z}{y+z}, -\frac{xz}{(y+z)^2}, \frac{xy}{(y+z)^2} \rangle$$

Gradient optimizes direction derivative

$\nabla f(p)$ direction realizes the maximum D_u ~~and~~ $a^< D_u f(p)$

$D_u f(p) = \nabla f(\vec{p}) \cdot \vec{u} = |\nabla f(p)||\vec{u}|\cos\theta = |\nabla f(p)|\cos\theta$

Ex. In which direction is $f(x,y,z) = \frac{xz}{y+z}$ attain is maximal directional derivative $P = \langle 1, 1, -2 \rangle$? : what is the max?

sol: $D_u f(p)$ is maximized in the direction of $\nabla f(p)$

$\nabla f = \langle \frac{z}{y+z}, -\frac{xz}{(y+z)^2}, \frac{xy}{(y+z)^2} \rangle$

$\nabla f(P) = \langle \frac{-2}{1-2}, -\frac{-2}{(1-2)^2}, \frac{1}{(1-2)^2} \rangle = \langle 2, 2, 1 \rangle$

$D_u f(p)$ is maximized in direction $\vec{u} = \frac{1}{3}\langle 2, 2, 1 \rangle$

and maximum $|\nabla f(p)| = 3$

Def: Let f be a function f has a maximum at P when $f(\vec{p}) \geq f(\vec{x})$ for all x nearby to $\vec{p}$. f has a global maximum value at $\vec{p}$ when $f(\vec{p}) = f(\vec{x})$.

Def: A critical point of function f is a point $\vec{p}$ such that either $\nabla f(p)$ does not exist or $\nabla f(p) = \vec{0}$

If f attains a local extrema at P then P is a critical point